

Parallel \ast -Ricci tensor of real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$

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ABSTRACT: In this paper the idea of studying real hypersurfaces in non-flat complex space forms, whose \ast -Ricci tensor satisfies geometric conditions is presented. More precisely, the non-existence of three dimensional real hypersurfaces in non-flat complex space forms with parallel \ast -Ricci tensor is proved. At the end of the paper ideas for further research on \ast -Ricci tensor are given.

Keywords: Real hypersurface, Parallel, \ast -Ricci tensor, Complex projective plane, Complex hyperbolic plane.

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1 Introduction

A *complex space form* is an n -dimensional Kaehler manifold of constant holomorphic sectional curvature c . A complete and simply connected complex space form is complex analytically isometric to

- a complex projective space $\mathbb{C}P^n$ if $c > 0$,
- a complex Euclidean space \mathbb{C}^n if $c = 0$,
- or a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$.

The symbol $M_n(c)$ is used to denote the complex projective space $\mathbb{C}P^n$ and complex hyperbolic space $\mathbb{C}H^n$, when it is not necessary to distinguish them. Furthermore, since $c \neq 0$ in previous two cases the notion of non-flat complex space form refers to both them.

Let M be a real hypersurface in a non-flat complex space form. An almost contact metric structure (φ, ξ, η, g) is defined on M induced from the Kaehler metric G and the complex structure J on $M_n(c)$. The *structure vector field* ξ is called *principal* if $A\xi = \alpha\xi$, where A is the shape operator of M and $\alpha = \eta(A\xi)$ is a smooth function. A real hypersurface is called *Hopf hypersurface*, if ξ is principal and α is called *Hopf principal curvature*.

The *Ricci tensor* S of a Riemannian manifold is a tensor field of type (1,1) and is given by

$$g(SX, Y) = \text{trace}\{Z \mapsto R(Z, X)Y\}.$$

If the Ricci tensor of a Riemannian manifold satisfies the relation

$$S = \lambda g,$$

where λ is a constant, then it is called *Einstein*.

Real hypersurfaces in non-flat complex space forms have been studied in terms of their Ricci tensor S , when it satisfies certain geometric conditions extensively. Different types of parallelism or invariance of the Ricci tensor are issues of great importance in the study of real hypersurfaces.

In [4] it was proved the non-existence of real hypersurfaces in non-flat complex space forms $M_n(c)$, $n \geq 3$ with parallel Ricci tensor, i.e. $(\nabla_X S)Y = 0$, for any $X, Y \in TM$. In [5] Kim extended the result of non-existence of real hypersurfaces with parallel Ricci tensor in case of three dimensional real hypersurfaces. Another type of parallelism which was studied is that of ξ -parallel Ricci tensor, i.e. $(\nabla_\xi S)Y = 0$ for any $Y \in TM$. More precisely in [6] Hopf hypersurfaces in non-flat complex space forms with constant mean curvature and ξ -parallel Ricci tensor were classified. More details on the study of Ricci tensor of real hypersurfaces are included in Section 6 of [7].

Motivated by Tachibana, who in [9] introduced the notion of $*$ -Ricci tensor on almost Hermitian manifolds, in [2] Hamada defined the $*$ -Ricci tensor of real hypersurfaces in non-flat complex space forms by

$$g(S^*X, Y) = \frac{1}{2}(\text{trace}\{\varphi \circ R(X, \varphi Y)\}), \quad \text{for } X, Y \in TM.$$

The $*$ -Ricci tensor S^* is a tensor field of type (1,1) defined on real hypersurfaces. Taking into account the work that so far has been done in the area of studying real hypersurfaces in non-flat complex space forms in terms of their tensor fields, the following issue raises naturally:

The study of real hypersurfaces in terms of their $$ -Ricci tensor S^* , when it satisfies certain geometric conditions.*

In this paper three dimensional real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ equipped with parallel $*$ -Ricci tensor are studied. Therefore, the following condition is satisfied

$$(\nabla_X S^*)Y = 0, \quad X, Y \in TM. \quad (1.1)$$

More precisely the following Theorem is proved.

Main Theorem: *There do not exist real hypersurfaces in $\mathbb{C}P^2$ and in $\mathbb{C}H^2$, whose $*$ -Ricci tensor is parallel.*

The paper is organized as follows: In Section 2 preliminaries relations for real hypersurfaces in non-flat complex space forms are presented. In Section 3 the proof of Main Theorem is provided. Finally, in Section 4 ideas for further research on the above issue are included.

2 Preliminaries

Throughout this paper all manifolds, vector fields etc are assumed to be of class C^∞ and all manifolds are assumed to be connected. Furthermore, the real hypersurfaces M are supposed to be without boundary.

Let M be a real hypersurface immersed in a non-flat complex space form $(M_n(c), G)$ with complex structure J of constant holomorphic sectional curvature c . Let N be a locally defined unit normal vector field on M and $\xi = -JN$ the structure vector field of M .

For a vector field X tangent to M the following relation holds

$$JX = \varphi X + \eta(X)N,$$

where φX and $\eta(X)N$ are the tangential and the normal component of JX respectively. The Riemannian connections $\bar{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M by

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

where g is the Riemannian metric induced from the metric G .

The shape operator A of the real hypersurface M in $M_n(c)$ with respect to N is given by

$$\overline{\nabla}_X N = -AX.$$

The real hypersurface M has an almost contact metric structure (φ, ξ, η, g) induced from the complex structure J on $M_n(c)$, where φ is the *structure tensor* and it is a tensor field of type (1,1). Moreover, η is an 1-form on M such that

$$g(\varphi X, Y) = G(JX, Y), \quad \eta(X) = g(X, \xi) = G(JX, N).$$

Furthermore, the following relations hold

$$\begin{aligned} \varphi^2 X &= -X + \eta(X)\xi, & \eta \circ \varphi &= 0, & \varphi \xi &= 0, & \eta(\xi) &= 1, \\ g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y), & g(X, \varphi Y) &= -g(\varphi X, Y). \end{aligned}$$

Since J is complex structure implies $\nabla J = 0$. The last relation leads to

$$\nabla_X \xi = \varphi AX, \quad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi. \quad (2.1)$$

The ambient space $M_n(c)$ is of constant holomorphic sectional curvature c and this results in the Gauss and Codazzi equations to be given respectively by

$$\begin{aligned} R(X, Y)Z &= \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X \\ &\quad - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z] + g(AY, Z)AX - g(AX, Z)AY, \\ (\nabla_X A)Y - (\nabla_Y A)X &= \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi] \end{aligned} \quad (2.2)$$

where R denotes the Riemannian curvature tensor on M and X, Y, Z are any vector fields on M .

The tangent space $T_P M$ at every point $P \in M$ can be decomposed as

$$T_P M = \text{span}\{\xi\} \oplus \mathbb{D},$$

where $\mathbb{D} = \ker \eta = \{X \in T_P M : \eta(X) = 0\}$ and is called *holomorphic distribution*. Due to the above decomposition the vector field $A\xi$ can be written

$$A\xi = \alpha\xi + \beta U, \quad (2.3)$$

where $\beta = |\varphi \nabla_\xi \xi|$ and $U = -\frac{1}{\beta} \varphi \nabla_\xi \xi \in \ker(\eta)$ provided that $\beta \neq 0$.

Since the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c following similar calculations to those in Theorem 2 in [3] and taking into account relation (2.2), it is proved that the *-Ricci tensor S^* of M is given by

$$S^* = -[\frac{cn}{2}\varphi^2 + (\varphi A)^2]. \quad (2.4)$$

3 Proof of Main Theorem

Let M be a non-Hopf hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, i.e. $M_2(c)$. Then the following relations hold on every non-Hopf three-dimensional real hypersurface in $M_2(c)$.

Lemma 3.1 *Let M be a real hypersurface in $M_2(c)$. Then the following relations hold on M*

$$AU = \gamma U + \delta \varphi U + \beta \xi, \quad A\varphi U = \delta U + \mu \varphi U, \quad (3.1)$$

$$\nabla_U \xi = -\delta U + \gamma \varphi U, \quad \nabla_{\varphi U} \xi = -\mu U + \delta \varphi U, \quad \nabla_\xi \xi = \beta \varphi U, \quad (3.2)$$

$$\nabla_U U = \kappa_1 \varphi U + \delta \xi, \quad \nabla_{\varphi U} U = \kappa_2 \varphi U + \mu \xi, \quad \nabla_\xi U = \kappa_3 \varphi U, \quad (3.3)$$

$$\nabla_U \varphi U = -\kappa_1 U - \gamma \xi, \quad \nabla_{\varphi U} \varphi U = -\kappa_2 U - \delta \xi, \quad \nabla_\xi \varphi U = -\kappa_3 U - \beta \xi, \quad (3.4)$$

where $\gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M and $\{U, \varphi U, \xi\}$ is an orthonormal basis of M .

For the proof of the above Lemma see [8]

Let M be a real hypersurface in $M_2(c)$, i.e. $\mathbb{C}P^2$ or $\mathbb{C}H^2$, whose *-Ricci tensor satisfies relation (1.1), which is more analytically written

$$\nabla_X(S^*Y) = S^*(\nabla_X Y), \quad X, Y \in TM. \quad (3.5)$$

We consider the open subset \mathcal{N} of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0, \text{ in a neighborhood of } P\}.$$

In what follows we work on the open subset \mathcal{N} .

On \mathcal{N} relation (2.3) and relations (3.1)-(3.4) of Lemma 3.1 hold. So relation (2.4) for $X \in \{U, \varphi U, \xi\}$ taking into account $n = 2$ and relations (2.3) and (3.1) yields

$$S^*\xi = \beta \mu U - \beta \delta \varphi U, \quad S^*U = (c + \gamma \mu - \delta^2)U \quad \text{and} \quad S^*\varphi U = (c + \gamma \mu - \delta^2)\varphi U. \quad (3.6)$$

The inner product of relation (3.5) for $X = Y = \xi$ with ξ due to the first and the third of (3.6), the first of (2.1) for $X = \xi$ and the third of relations (3.3) and (3.4) implies

$$\delta = 0. \quad (3.7)$$

Moreover, the inner product of relation (3.5) for $X = \varphi U$ and $Y = \xi$ with ξ because of (3.7), the first of (2.1) for $X = \varphi U$, the first and the second of (3.6) and the second of (3.3) results in

$$\mu = 0.$$

Finally, the inner product of relation (3.5) for $X = \xi$ and $Y = \varphi U$ with ξ taking into account $\mu = \delta = 0$, the first and the third of (3.6) and the last relation of (3.4) leads to

$$c = 0,$$

which is a contradiction. So the open subset \mathcal{N} is empty and we lead to the following Proposition.

Proposition 3.2 *Every real hypersurface in $M_2(c)$ whose *-Ricci tensor is parallel, is a Hopf hypersurface.*

Since M is a Hopf hypersurface, the structure vector field ξ is an eigenvector of the shape operator, i.e. $A\xi = \alpha\xi$. Due to Theorem 2.1 in [7] α is constant. We consider a point $P \in M$ and choose a unit principal vector field $W \in \mathbb{D}$ at P , such that $AW = \lambda W$ and $A\varphi W = \nu\varphi W$. Then $\{W, \varphi W, \xi\}$ is a local orthonormal basis and the following relation holds (Corollary 2.3 [7])

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \quad (3.8)$$

The first of relation (2.1) and relation (2.4) for $X \in \{W, \varphi W, \xi\}$ because of $A\xi = \alpha\xi$, $AW = \lambda W$ and $A\varphi W = \nu\varphi W$ implies respectively

$$\nabla_W \xi = \lambda\varphi W \quad \text{and} \quad \nabla_{\varphi W} \xi = -\nu W \quad (3.9)$$

$$S^*\xi = 0, \quad S^*W = (c + \lambda\nu)W \quad \text{and} \quad S^*\varphi W = (c + \lambda\nu)\varphi W. \quad (3.10)$$

Relation (3.5) for $X = W$ and $Y = \xi$ because of the first of (3.9) and the first and third relation of (3.10) yields

$$\lambda(c + \lambda\nu) = 0.$$

Suppose that $(c + \lambda\nu) \neq 0$ then the above relation results in $\lambda = 0$. Moreover, relation (3.5) for $X = \varphi W$ and $Y = \xi$ because of the second of (3.9) and the first and second relation of (3.10) yields

$$\nu = 0.$$

Substitution of $\lambda = \nu = 0$ in (3.8) results in $c = 0$, which is a contradiction. So relation $c = -\lambda\nu$ holds. The last one implies $\lambda\nu \neq 0$ since $c \neq 0$.

Suppose that $\lambda = -\frac{c}{\nu}$. Substitution of the last one in (3.8) leads to

$$2\alpha\nu^2 + 5c\nu - 2\alpha c = 0. \quad (3.11)$$

In case of $\mathbb{C}P^2$ we have that $c = 4$ and from equation (3.11) there is always a solution for ν . So ν is constant and λ will be also constant. Therefore, the real hypersurface has three distinct constant eigenvalues. The latter results in M being a real hypersurface of type (B), i.e. a tube of radius r over complex quadric. Substitution of the eigenvalues of type (B) in $\lambda\nu = -c$ leads to a contradiction. So no real hypersurface in $\mathbb{C}P^2$ has parallel *-Ricci tensor (eigenvalues can be found in [7]).

In case of $\mathbb{C}H^2$ we have that $c = -4$ and from equation (3.11) there is a solution for ν if $0 \leq \alpha^2 \leq \frac{25}{4}$. If $\alpha = 0$ equation (3.11) implies $c\nu = 0$, which is impossible. So there is a solution for ν if $0 < \alpha^2 \leq \frac{25}{4}$ and ν will be constant. The latter results in that λ is also constant and so the real hypersurface is of type (B), i.e. a tube of radius r around totally geodesic $\mathbb{R}H^n$. Substitution of the eigenvalues of type (B) in $\lambda\nu = -c$ leads to a contradiction and this completes the proof of our Main Theorem (eigenvalues can be found in [1]).

4 Discussion-Open Problems

In this paper three dimensional real hypersurfaces in non-flat complex space forms with parallel *-Ricci tensor are studied and the non-existence of them is proved. Therefore, a question which raises in a natural way is

Are there real hypersurfaces in non-flat complex space forms of dimension greater than three with

parallel \ast -Ricci tensor?

Generally, the next step in the study of real hypersurfaces in non-flat complex space forms is to study them when a tensor field P type $(1,1)$ of them satisfies other types of parallelism such as the \mathbb{D} -parallelism or ξ -parallelism. The first one implies that P is parallel in the direction of any vector field X orthogonal to ξ , i.e. $(\nabla_X P)Y = 0$, for any $X \in \mathbb{D}$, and the second one implies that P is parallel in the direction of the structure vector ξ , i.e. $(\nabla_\xi P)Y = 0$. So the questions which should be answered are the following

Are there real hypersurfaces in non-flat complex space forms whose \ast -Ricci tensor satisfies the condition of \mathbb{D} -parallelism or ξ -parallelism?

Finally, other types of parallelism play important role in the study of real hypersurfaces is that of semi-parallelism and pseudo-parallelism. A tensor field P of type $(1, s)$ is said to be *semi-parallel* if it satisfies $R \cdot P = 0$, where R is the Riemannian curvature tensor and acts as a derivation on P . Moreover, P is said to be *pseudo-parallel* if there exists a function L such that $R(X, Y) \cdot P = L\{(X \wedge Y) \cdot P\}$, where $(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y$. So the questions are:

Are there real hypersurfaces in non-flat complex space forms with semi-parallel or pseudo-parallel \ast -Ricci tensor?

The importance of answering the above question lays in the fact that the class of real hypersurfaces with parallel \ast -Ricci tensor is included in the class of real hypersurfaces with semi-parallel \ast -Ricci tensor. Furthermore, the last one is included in the class of real hypersurfaces with pseudo-parallel \ast -Ricci tensor.

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